1) $z = e^{2j\varphi}$

(1)

2) $z = e^{2j\varphi} = \cos(2\varphi) + j\sin(2\varphi)$

D'autre part

$$z = \frac{\cos(\varphi) + j\sin(\varphi)}{\cos(\varphi) - j\sin(\varphi)} = \frac{(\cos(\varphi) + j\sin(\varphi))(\cos\varphi + j\sin(\varphi))}{\cos^2(\varphi) + \sin^2(\varphi)}$$

Donc $z = (\cos^2\varphi - \sin^2\varphi) + j \times (2\sin\varphi\cos\varphi)$

Donc $\boxed{\cos(2\varphi) = \cos^2(\varphi) - \sin^2(\varphi)}$

et $\boxed{\sin(2\varphi) = 2\sin(\varphi)\cos(\varphi).}$

$$\tan(2\varphi) = \frac{\sin(2\varphi)}{\cos(2\varphi)} = \frac{2\sin(\varphi)\cos(\varphi)}{\cos^2(\varphi) - \sin^2(\varphi)} = \frac{\cos^2(\varphi)\left(2\frac{\sin(\varphi)}{\cos(\varphi)}\right)}{\cos^2(\varphi)\left(1 - \frac{\sin^2(\varphi)}{\cos^2(\varphi)}\right)}$$

Donc $\tan(2\varphi) = \dfrac{2\tan(\varphi)}{1 - \tan^2(\varphi)} = \boxed{\dfrac{2\tan(\varphi)}{1 - \tan^2(\varphi)}}$

3) $$I = \int_0^{\pi/4} \frac{1}{\cos(x)}\,dx = \int_0^{\pi/4} \frac{1}{\cos^2(x/2) - \sin^2(x/2)}\,dx$$

$$I = \int_0^{\pi/4} \frac{1}{\cos^2(\frac{x}{2})\left(1 - \tan^2(\frac{x}{2})\right)}\,dx$$

On pose $u = \tan(x/2)$, donc $du = \frac{1}{2} \times \frac{1}{\cos^2(x/2)}\,dx$

Donc $$I = \int_0^{\tan(\pi/8)} \frac{2\,du}{1 - u^2} = \int_0^{\tan(\pi/8)} \left(\frac{1}{1+u} + \frac{1}{1-u}\right) du$$

$$I = \int_0^{\tan(\pi/8)} \frac{du}{1+u} - \int_0^{\tan(\pi/8)} \frac{du}{u-1} = \left[\ln(1+u)\right]_0^{\tan(\pi/8)} - \left[\ln|u-1|\right]_0^{\tan(\frac{\pi}{8})}$$

$$I = \ln(1 + \tan(\pi/8)) - \ln((1 - \tan(\pi/8))) = \ln\left(\frac{1 + \tan(\pi/8)}{1 - \tan(\pi/8)}\right) \simeq 0{,}88$$

Exercice 2

1) $|1+iz| = |1-iz|$ (1)

on pose $z = x+iy$, donc $1+iz = 1+i(x+iy) = (1-y)+ix$

et $1-iz = 1-i(x+iy) = (1+y)-ix$

(1) $\Leftrightarrow (1-y)^2+x^2 = (1+y)^2+x^2 \Leftrightarrow 1+y^2-2y+x^2 = 1+2y+y^2+x^2$

$\Leftrightarrow y=0$

Donc l'ensemble des nombres complexes qui vérifient (1) sont les nombres réels

soit $\mathbb{R}$.

2) on pose $z = \tan(\alpha)$ et $a = \tan(\beta)$

Donc $\dfrac{1+iz}{1-iz} = \dfrac{1+i\tan(\alpha)}{1-i\tan(\alpha)} = e^{2i\alpha}$

$\dfrac{1+ja}{1-ja} = \dfrac{1+j\tan(\beta)}{1-j\tan(\beta)} = e^{2j\beta}$

Donc $\left(\dfrac{1+iz}{1-iz}\right)^m = \dfrac{1+ja}{1-ja} \Leftrightarrow \left(e^{2i\alpha}\right)^m = e^{2j\beta}$

$\Leftrightarrow e^{2i\alpha} = e^{2j\beta/m} \Leftrightarrow 2i\alpha = 2j\beta/m$

Donc $\alpha = \beta/m$, donc $\mathrm{Arctan}(z) = \beta/m$

Donc $z = \tan\left(\dfrac{\mathrm{Arctan}(a)}{m}\right)$.