$$\begin{pmatrix} (p-1) & -2 & 2 \\ 2 & p+3 & -2 \\ -2 & -2 & p+3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{p}{p^2+1} \\ \frac{1}{p^2+1} \\ 1 \end{pmatrix}$$

Soit A la matrice $\begin{pmatrix} (p-1) & -2 & 2 \\ 2 & p+3 & -2 \\ -2 & -2 & p+3 \end{pmatrix}$

Calculons ce déterminant

$$\operatorname{Det} A = (p-1)\begin{vmatrix} p+3 & -2 \\ -2 & p+3 \end{vmatrix} + 2\begin{vmatrix} 2 & -2 \\ -2 & p+3 \end{vmatrix} + 2\begin{vmatrix} 2 & p+3 \\ -2 & -2 \end{vmatrix}$$

$$= (p-1)\left(p^2+6p+9-4\right) + 2\left(2p+6-4\right) + 2\left(-4+2p+6\right)$$

$$= (p-1)\left(p^2+6p+5\right) + 4p+4+4p+4$$

$$= p^3+6p^2+5p-p^2-6p-5+8p+8$$

$$= p^3+5p^2+7p+3$$

$$= (p+1)(p+3)(p+1) = (p+3)(p+1)^2$$

$p \neq -3$ et $p \neq -1$, donc $\operatorname{Det} A \neq 0$, donc la matrice A est inversible.

Calculons la matrice des cofacteurs de A

$$B = \begin{pmatrix} \begin{vmatrix} p+3 & -2 \\ -2 & p+3 \end{vmatrix} & -\begin{vmatrix} 2 & -2 \\ -2 & p+3 \end{vmatrix} & \begin{vmatrix} 2 & p+3 \\ -2 & -2 \end{vmatrix} \\ -\begin{vmatrix} -2 & 2 \\ -2 & p+3 \end{vmatrix} & \begin{vmatrix} p-1 & 2 \\ -2 & p+3 \end{vmatrix} & -\begin{vmatrix} p-1 & -2 \\ -2 & -2 \end{vmatrix} \\ \begin{vmatrix} -2 & 2 \\ p+3 & -2 \end{vmatrix} & -\begin{vmatrix} p-1 & 2 \\ 2 & -2 \end{vmatrix} & \begin{vmatrix} p-1 & -2 \\ 2 & p+3 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} p^2+6p+5 & -2p-2 & 2p+2 \\ 2p+2 & (p+1)^2 & 2p+2 \\ -2-2p & 2p+2 & (p+1)^2 \end{pmatrix}$$

Transposons B pour obtenir C

$$C = \begin{pmatrix} (p+1)(p+5) & 2(p+1) & -2(p+1) \\ -2(p+1) & (p+1)^2 & 2(p+1) \\ 2(p+1) & 2(p+1) & (p+1)^2 \end{pmatrix}$$

Donc la matrice $A^{-1} = \dfrac{C}{\det A} = \begin{pmatrix} \dfrac{(p+1)(p+5)}{(p+1)^2(p+3)} & \dfrac{2(p+1)}{(p+1)^2(p+3)} & \dfrac{-2(p+1)}{(p+1)^2(p+3)} \\ \dfrac{-2(p+1)}{(p+1)^2(p+3)} & \dfrac{(p+1)^2}{(p+1)^2(p+3)} & \dfrac{2(p+1)}{(p+1)^2(p+3)} \\ \dfrac{2(p+1)}{(p+1)^2(p+3)} & \dfrac{2(p+1)}{(p+1)^2(p+3)} & \dfrac{(p+1)^2}{(p+1)^2(p+3)} \end{pmatrix}$

Donc $A^{-1} = \begin{pmatrix} \dfrac{p+5}{(p+1)(p+3)} & \dfrac{2}{(p+1)(p+3)} & \dfrac{-2}{(p+1)(p+3)} \\ \dfrac{-2}{(p+1)(p+3)} & \dfrac{1}{(p+3)} & \dfrac{2}{(p+1)(p+3)} \\ \dfrac{2}{(p+1)(p+3)} & \dfrac{2}{(p+1)(p+3)} & \dfrac{1}{(p+3)} \end{pmatrix}$

Donc $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \begin{pmatrix} \dfrac{p}{p^2+1} \\ \dfrac{1}{p^2+1} \\ 1 \end{pmatrix}$

Donc $x = \left(\dfrac{(p+5)}{(p+1)(p+3)} * \dfrac{p}{p^2+1}\right) + \left(\dfrac{2}{(p+1)(p+3)} \times \dfrac{1}{p^2+1}\right) + \left(\dfrac{-2}{(p+1)(p+3)} \times 1\right)$

$= \dfrac{p^2+5p+2}{(p+1)(p+3)(p^2+1)} - \dfrac{2(p^2+1)}{(p+1)(p+3)(p^2+1)} = \dfrac{p^2+5p+2-2p^2-2}{(p^2+1)(p+1)(p+3)}$

$= \dfrac{p(5-p)}{(p^2+1)(p+1)(p+3)}$