Exercice 1

a) $y' + \frac{y}{x^2} = -\frac{1}{x^3}$

$$\mu y' + \frac{\mu y}{x^2} = \mu \times (-1/x^3) \quad (E)$$

$$\mu' = \frac{\mu}{x^2} \Leftrightarrow \frac{\mu'}{\mu} = \frac{1}{x^2} \Leftrightarrow \ln(\mu) = -\frac{1}{x}$$

$$\Leftrightarrow \mu = e^{-1/x}.$$

on a donc $y e^{-1/x} + \frac{1}{x^2} e^{-1/x} y = -\frac{1}{x^3} \times e^{-1/x}$

$$\Rightarrow \left[y e^{-1/x} \right]' = -\frac{1}{x^3} e^{-1/x}$$

$$\Rightarrow y = \left[\left[-\int \frac{1}{x^3} e^{-1/x} \right] \times e^{1/x} \right] + K e^{1/x}$$

b) $\mu y' + \tan(x) \mu y = \cos^2 x \, \mu \quad (E)$

t $\mu' = \mu \tan(x) \Leftrightarrow \frac{\mu'}{\mu} = \tan(x) \Rightarrow \ln(\mu) = -\ln \cos x$

$$\Rightarrow \mu(x) = \frac{1}{\cos x}$$

$$(E) \Rightarrow \frac{1}{\cos x} y' + \frac{\sin x}{\cos^2 x} y = \cos x$$

$$\left[\frac{y}{\cos x} \right]' = \cos x \Rightarrow y/\cos x = \sin x + K$$

$$\Rightarrow \underline{\underline{y = \sin x \cos x + K \sin x}}$$

c) $\mu(1+x^2) y' - 4\mu x y = \mu(3x^2 - x^4) \quad (E)$

$$\cancel{[\mu(1+x^2)]' = -4\mu x} \quad \cancel{\Rightarrow 2x\mu + \mu'(1+x^2) = -4\mu x}$$

$$\cancel{\Leftrightarrow \mu'(1+x^2) = -6\mu x} \quad \cancel{\Leftrightarrow \frac{\mu'}{\mu} = -\frac{6x}{1+x^2}} \quad \cancel{\Rightarrow \ln(\mu) = -3\ln(1+x^2)}$$

on trouve une soluti sans second membre.

$$(1+x^2)y' - 4xy = 0 \Leftrightarrow \frac{y'}{y} = \frac{4x}{1+x^2} \Rightarrow \ln(y) = 2\ln(1+x^2)$$
$$\Rightarrow y = (1+x^2)^2$$

on essaye de trouver une soluti particulière de la forme ax^3+bx^2+cx+d.

$$(1+x^2)(3ax^2+2bx+c) - 4x(ax^3+bx^2+cx+d) = 3x^2-x^4$$

$$\Leftrightarrow \cancel{3ax^2}+2b\cancel{x}+c+3\cancel{ax^4}+2\cancel{bx^3}+\cancel{cx^2}-4\cancel{ax^4}-4\cancel{bx^3}-4\cancel{cx^2}-4dx = 3x^2-x^4$$

$\Leftrightarrow$ ~~$a=0$~~ ~~$2b-4b$~~

$$\Leftrightarrow x^4(-a) + x^3(-2b) + x^2(3a+c-4c) + x(2b-4d) + c = 3x^2-x^4$$

$a=1$ $\quad b=0 \quad 3+-3c=3 \quad c=0 \quad d=0 \quad c=0$

soluti particulière $\quad y = x^3$

Donc $y = (1+x^2)^2 + x^3$

d) $\quad y'+2y = e^x \quad (E)$

$$uy'+2uy = ue^x \qquad [u'] = 2u \Rightarrow \frac{u'}{u} = 2 \qquad \ln(u) = 2x$$
$$\Rightarrow u = e^{2x}$$

$$(E) \Leftrightarrow e^{2x}y' + 2e^{2x}y = e^{3x}$$
$$\Leftrightarrow [ye^{2x}]' = e^{3x} \Leftrightarrow ye^{2x} = \frac{1}{3}e^{3x} + k$$
$$\Leftrightarrow \underline{\underline{y = \frac{1}{3}e^x + ke^{-2x}}}$$

e) $\quad y' - \frac{1+x^2}{x(x^2-1)}y = 2$

$$uy' - u\frac{1+x^2}{x(x^2-1)}y = 2 \qquad u' = \frac{-u(1+x^2)}{x(x^2-1)}$$

$$\frac{u'}{u} = \frac{1+x^2}{x(1-x^2)} = \frac{1+x^2}{x(1-x)(1+x)}$$

cherchons a, b, c tel que $\frac{1+x^2}{x(1-x)(1+x)} = \frac{a}{x} + \frac{b}{1-x} + \frac{c}{1+x}$

$$\frac{a}{x}+\frac{b}{1-x}+\frac{c}{1+x}=\frac{a(1-x)(1+x)+bx(1+x)+c(x)(1-x)}{x(1-x)(1+x)}$$

$$=\frac{a-ax^2+bx+bx^2+cx-cx^2}{x(1-x)(1+x)}=\frac{x^2(b-a-c)+x(b+c)+a}{x(1-x)(1+x)}$$

Dc ~~$a=1$, $b-a=1 \Rightarrow b=0$, $b+c=0$~~

$$\begin{cases} b-a-c=1 \\ b+c=0 \\ a=1 \end{cases} \Rightarrow \begin{cases} b=-c \\ 2b=2 \\ a=1 \end{cases} \Leftrightarrow \begin{cases} a=1 \\ b=1 \\ c=-1 \end{cases}$$

Dc $\dfrac{1+x^2}{x(1-x)(1+x)}=\dfrac{1}{x}+\dfrac{1}{1-x}-\dfrac{1}{1+x}$.

Dc $\dfrac{u'}{u}=\dfrac{1}{x}+\dfrac{1}{1-x}-\dfrac{1}{1+x} \Rightarrow \ln(u)=\ln|x|-\ln|1-x|-\ln|1+x|$

$$u=\frac{x}{x^2-1}$$

Dc (E) $\Leftrightarrow \dfrac{xy'}{x^2-1}-\dfrac{1+x^2}{(x^2-1)^2}y=2$

$\Leftrightarrow \left[\dfrac{xy}{x^2-1}\right]'=2 \Leftrightarrow \dfrac{xy}{x^2-1}=2x+K.$

$\Leftrightarrow y=2(x^2-1)+K\dfrac{x^2-1}{x}=2x^2-2+Kx-\dfrac{K}{x}$

f) $y'+\dfrac{y}{\sqrt{1+x^2}}=1-\dfrac{1}{\sqrt{1+x^2}}$

$uy'+\dfrac{uy}{\sqrt{1+x^2}}=\left(1-\dfrac{1}{\sqrt{1+x^2}}\right)u$

$u'=\dfrac{u}{\sqrt{1+x^2}} \Leftrightarrow \dfrac{u'}{u}=\dfrac{1}{\sqrt{1+x^2}} \Rightarrow \ln(u)=\operatorname{Argsh}(x)$

$u=e^{\operatorname{Argsh}(x)}$

$$\left[y\, e^{\operatorname{Argsh}(x)}\right]' = \left(1 - \frac{1}{\sqrt{1+x^2}}\right) e^{\operatorname{Argsh}(x)}$$

$$y\, e^{\operatorname{Argsh}(x)} = \int e^{\operatorname{Argsh}(x)} - e^{\operatorname{Argsh}(x)}$$

$$\boxed{y = \frac{\int e^{\operatorname{Argsh}(x)}}{e^{\operatorname{Argsh}(x)}} - 1}$$

g) $y' + 2y = xe^{x} + e^{-2x}$

$uy' + 2uy = uxe^{x} + ue^{-2x} \quad (E)$

$u' = 2u \qquad \frac{u'}{u} = 2 \qquad \ln(u) = 2x \qquad u = e^{2x}$

$e^{2x}y' + 2e^{2x}y = xe^{3x} + 1$

$[ye^{2x}]' = xe^{3x} + 1$

$ye^{2x} = \int (xe^{3x} + 1)\,dx$

$ye^{2x} = \left[x\frac{e^{3x}}{3}\right] - \frac{1}{3}e^{3x} + x + K$

$ye^{2x} = \frac{1}{3}(x-1)e^{3x} + x + K_1$

$$\underline{\underline{y = \frac{1}{3}(x-1)e^{x} + xe^{-2x} + Ke^{-2x}}}$$

h) $y' - y = \sin x \qquad uy' - uy = \sin x$

$u' = -u \Rightarrow \frac{u'}{u} = -1 \qquad \ln(u) = -x \Rightarrow u = e^{-x}$

$-e^{-x}y' + e^{-x}y = \sin x\, e^{-x} \Leftrightarrow [e^{-x}y]' = \sin x\, e^{-x}$

$e^{-x}y = -\frac{1}{2}(\cos x + \sin x)e^{-x} + K$

$$\boxed{y = -\frac{1}{2}\cos x - \frac{1}{2}\sin x + Ke^{x}}$$

1) $xy' + y = x^2 \ln(x)$

$\mu x y' + \mu y = \mu x^2 \ln(x)$

$[\mu x]' = \mu \quad \Leftrightarrow \quad \mu' x + \mu = \mu \quad \Leftrightarrow \quad \mu' = 0 \quad \Rightarrow \mu = K$

$[xy]' = x^2 \ln(x)$

$xy = \frac{x^3}{3}\ln(x) - \frac{x^3}{9} + K.$

$$\boxed{y = \frac{x^2}{3}\ln(x) - \frac{x^2}{9} + \frac{K}{x}}$$