Ex 8

3) $M \in E_1(A) \Leftrightarrow AM = M \Leftrightarrow AM - M = 0$

$\Leftrightarrow AM - IM = 0 \Leftrightarrow (A-I)M = 0$

Comme $A-I$ inversible, $\exists B$ tel que $(A-I)^{-1} = B$

Donc si $(A-I)M = 0 \Leftrightarrow B \times (A-I) \times M = 0 \Leftrightarrow M = 0$ CQFD.

4) $A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

on cherche M tel que $AM = M$

on pose $M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

or $A - I = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

$\det(A-I) = 4 \neq 0$ Donc $A-I$ inversible donc $\underline{\underline{M = 0}}$

on cherche M tel que $A^2 M = AM$

$\Leftrightarrow A^2M - AM = 0 \Leftrightarrow AAM - AM = 0 \Leftrightarrow A(A-I)M = 0$

$\det A \neq 0$ et $\det(A-I) \neq 0$ Donc $\underline{\underline{M = 0}}$

5) $B = \begin{pmatrix} 3 & -2 & -1 \\ 1 & 0 & -1 \\ 2 & -2 & 0 \end{pmatrix}$ $P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

a) $\det P = 1 \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = -1 \neq 0$ Donc P est inversible

$P^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ -1 & 1 & 1 \end{pmatrix}$

b) $D = P^{-1} B P = \begin{pmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & -2 & -1 \\ 1 & 0 & -1 \\ 2 & -2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 2 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

c) $N = P^{-1}M \iff PN = P \times P^{-1} \times M$

$\iff PN = M \iff \underline{\underline{M = PN}}$

d) $M \in E_1(B) \iff BM = M \iff BPN = PN$

$\iff \underbrace{P^{-1}BP}\, N = P^{-1}P\,\cancel{N}$

$\iff DN = N \iff N \in E_1(D)$

e) $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ D n'est pas inversible

$D - I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ D - I n'est pas inversible

on pose $\cancel{\phi}\ N = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

$DN = N \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

$\iff \begin{pmatrix} a & b & c \\ 2d & 2e & 2f \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \iff \begin{cases} a = a \\ b = b \\ c = c \\ d = 0 \\ e = 0 \\ f = 0 \\ g = 0 \\ h = 0 \\ i = 0 \end{cases}$

Donc $N = \begin{pmatrix} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

f) $M = PN = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b & c \\ b & b & c \\ 0 & 0 & 0 \end{pmatrix}$